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# VIBRATIONS AND STABILITY OF SHELLS DUE TO LOADS AND TEMPERATURE*

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**Abstract**—The vibrations and stability of a thin cylindrical circular shell and a three-layered plate are studied. The structure is considered as a system with many degrees of freedom. This model permits investigations of the dynamic properties of the structure due to loads and temperature. An approximate solution of the wave problem is obtained.

## INTRODUCTION

THE general theory of shells is highly developed in the USSR by the scientific works of Vlassov [1], Goldenveizer [2], Novozhilov [3] and others. As a result, we can now effect the complicated design of many types of shells subjected to loads and temperature. But some dynamic and stability problems are not yet solved, for example the problem of dynamic temperature action. An approximate solution of these problems is worked out by taking into account the analogy existing between the differential equations of motion of a shell and those of a beam or plate on an elastic foundation. This mechanical model was studied by N. J. Hoff [4] and also by C. R. Steele [7] who studied the general equations of shell theory obtained by Novozhilov and had pointed out that the elastic foundation analogy is useful.

## STATEMENT OF PROBLEM

Hoff's model may be made more complicated by placing more masses and strings. The system now should have many degrees of freedom and in the limit case should be a beam on an elastic foundation. The solution of this dynamic model acted on by loads and temperature is more complicated. For this purpose we shall consider the three-layered system, which consists of two thin plates connected by elastic strings. The differential equations of motion for each of these plates are:

$$D_1\nabla^4 W_1 - N_1 + \mu\frac{\partial^2 W_1}{\partial t^2} = 0; \qquad D_2\nabla^4 W_2 + N_1 + \Delta N_1 + \mu_2\frac{\partial^2 W_2}{\partial t^2} = -\frac{1}{1-v}\nabla^2 W_T. \quad (1)$$

The thermal field is applied to one plate only. The force $N_1$ is the reaction in the string, and $\Delta N_1$ is the inertial force acting on the string. If an external load is applied too, it must be added to the right-hand side of equation (1).

The plate is on an elastic foundation if equal fields are applied together to both plates. Now the two equations (1) convert into a single equation

$$D\nabla^4 W + \left(\mu + \frac{h_0\mu_0}{6}\right)\frac{\partial^2 W}{\partial t^2} + \frac{2D_3}{h_0}W = -\frac{1}{2}\frac{1}{1-v}\nabla^2 W_T. \quad (2)$$

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To solve this equation we substitute:

$$W_i = W_{si}+W_{di} \tag{3}$$

where $W_{si}$ is the deflection due to the temperature acting statically and $W_{di}$ is the dynamic deflection due to inertial forces. Equation (2) for the determination of $W_{di}$ now transforms into the following equation:

$$D\nabla^4 W_{di}+\left(\mu+\frac{\mu_0 h_0}{6}\right)\ddot{W}_{di}+\frac{2D_3}{h_0}W_{di} = -\left(\mu+\frac{\mu_0 h_0}{6}\right)\ddot{W}_{si}. \tag{4}$$

The solution of equation (4) is:

$$W_{si} = \sum\sum (K_{nm})_i \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b} \tag{5}$$

$$W_{di} = \sum\sum [q^{(t)}]_i \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b} \tag{6}$$

where $(K_{nm})$ is

$$(K_{nm})_i = -\frac{16\mu_T(-1)^{(m+n)/2}\left[\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2\right]}{2(1-v)\pi^2 D\left\{nm\left[\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2\right]^2+\frac{2D_3}{h_0}\right\}}. \tag{7}$$

The determination of $q_{nm}(t)_i$ as a function of time is effected by solving the classical equation of second degree. For a suddenly applied temperature or heat flux this was done by H. Parkus

$$(q_{nm})_i = \frac{q}{2\lambda}(K_{nm})_i\left\{\frac{12\beta}{\pi^2(\omega_{nm})_i}\sin(\omega_{nm})_i t-\frac{96\beta^2}{\pi^4}\sum\frac{1}{j^4\beta^2(\omega_{nm})_i}\right.$$
$$\left.\times\left[\cos(\omega_{nm})_i t+\frac{(\omega_{nm})_i}{\beta}\frac{1}{j^2}\sin(\omega_{nm})_i t-\exp(-j\beta t)\right]\right\}. \tag{8}$$

This function will be more complicated for an impulse of temperature.

The frequencies $(\omega_{nm})_i$ were calculated by formula:

$$(\omega_{nm}^2)_i = \left\{\left[\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2\right]^2+\frac{2D_3}{h_0 D}\right\}\frac{D}{(\mu+\frac{1}{6}\mu_0 h)}. \tag{9}$$

The solution obtained may be applied to the dynamic problem of designing the circular cylindrical shell acted on by loads and temperature.

The equations of motion of cylindrical shell given by S. P. Timoshenko [9] and used by I. P. Jones and P. G. Bhuta [10] are the following:

$$\frac{Eh}{1-v^2}\left(\frac{\partial^2 u}{\partial x^2}-\frac{v\partial w}{a\partial x}\right) = \rho h\frac{\partial^2 u}{\partial t^2} \tag{10}$$

$$\frac{aEh^3}{12(1-v^2)}\frac{\partial^4 w}{\partial x^4}+\frac{Eh}{1-v^2}\left(\frac{w}{a}-v\frac{\partial u}{\partial x}\right) = pa-\rho ha\frac{\partial^2 w}{\partial t^2}. \tag{11}$$

For a long cylindrical shell subjected to a suddenly applied temperature uniformly distributed along its length the displacements in the direction of the shell axis are equal to zero and the two coupled equations (10) and (11) will be separated. Comparing equation (11) with equation (2) we find that the solution obtained for equation (2) is also the solution of equation (11).

## STABILITY

The vibrations and the stability problems of the shell are closely connected. The solution obtained permits an investigation of the stability of the shell and a determination of the dynamic critical load when the temperature is applied. The approximate solution given by Hoff shows that the critical load is highly dependent on the reaction force of the string which is acting on the concentrated mass placed in the middle of the shell length. In our problem the reactions of the elastic foundation are continuously distributed along the length of the shell. The determination of these reactions may be effected by formulae (3, 5 and 6), but the calculations involved are very extensive. For example, to calculate the displacement of one point of the shell the infinite series must be evaluated three times. This was done by using the electronic computer Ural II and the full diagrams of reactions were obtained (Fig. 1). The reactions are functions of time, and their values depend on the ratio

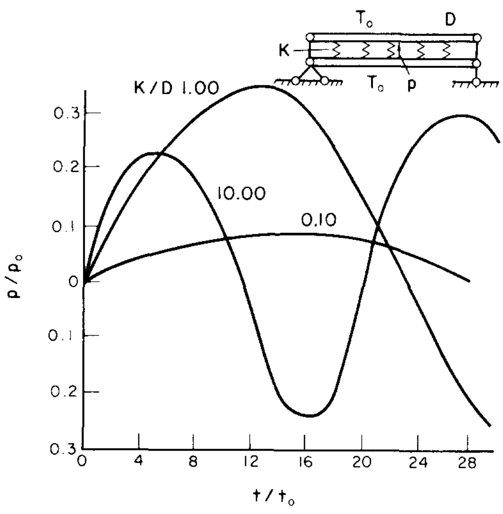


FIG. 1. The diagram of reaction in springs.

of rigidity of the shell and that of the generalized elastic foundation. It is interesting to note that the reaction has a maximum which corresponds to a certain value of diameter of the shell and its thickness. If the diameter of the shell is constant and the thickness could be changed, then the generalized reaction in the middle of the shell span is at first small (the curve $K/D = 0{\cdot}1$ in Fig. 1) then it becomes maximum (the curve $K/D = 1{\cdot}00$ in Fig. 1) and lastly it is small again (the curve $K/D = 10$ in Fig. 1). This conclusion is important to determination of the optimum value of the shell thickness corresponding to a given diameter.

## WAVE EFFECT

The equation of motion (10), if it is separated from equation (11), describes the wave propagation in a rod. This equation may be used for examination of the thermoelastic wave propagation along the length of cylindrical shell, when the temperature rises at its end. It is interesting to study the stress wave propagation, taking into account the coupling which exists between the equation of motion and heat conduction

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - \alpha\frac{\partial T}{\partial x}; \tag{12}$$

$$\rho c_\varepsilon[1+2\varepsilon(1-v)]\frac{\partial T}{\partial t} + \alpha T_0 E\frac{\partial^2 u}{\partial x \partial t} = \lambda\frac{\partial^2 T}{\partial x^2}. \tag{13}$$

These equations were studied by P. Chadwik [11], and solution of these equations is presented in the form:

$$u = u_s + u_d. \tag{14}$$

After the necessary transformations are carried out and the secondary terms are omitted, the equation of wave propagation will be:

$$\frac{1}{(c')^2}\frac{\partial^2 u_d}{\partial t^2} = \frac{\partial^2 u_d}{\partial x^2} \tag{15}$$

$c'$ is the modified wave velocity.

$$(c')^2 = \left[1 + \frac{\varepsilon(1-v)(1-2v)}{(1+v)[1-2\varepsilon(1-v)]}\right]c^2 \tag{16}$$

$\varepsilon$ is the coupling parameter, and $v$ Poisson's ratio. For example, we shall take the following temperature function:

$$T(x, t) = ate^{-bx} \tag{17}$$

The solution of equation (15) is:

$$u_d = A[\exp(-b(x-c't)) - \exp(-b(x+c't))]. \tag{18}$$

The maximum dynamic stresses are:

$$\sigma_{\max} = -0{\cdot}5\frac{E\alpha a}{bc'}[1-\exp(-2bc't)]. \tag{19}$$

The diagram of stress wave propagation in a cylindrical shell is plotted in Fig. 2 and some important conclusions may be reached. The modified stress wave arrives at a given point of the shell earlier than the longitudinal wave, but has a smaller amplitude. The stresses have a maximum in a cross section of the shell which is situated at a distance from the end of the shell. If the stresses should be so great that plastic deformation occurs then the plastic hinge will arise at some distance from the end of the shell. Hoff's model may be used to determine the stability of the shell. When the dynamic load and the temperature are acting together the thermoelastic stress wave propagates ahead of the longitudinal wave caused by a suddenly applied force and the stresses cannot be added. The experiments carried out on models confirm these conclusions.

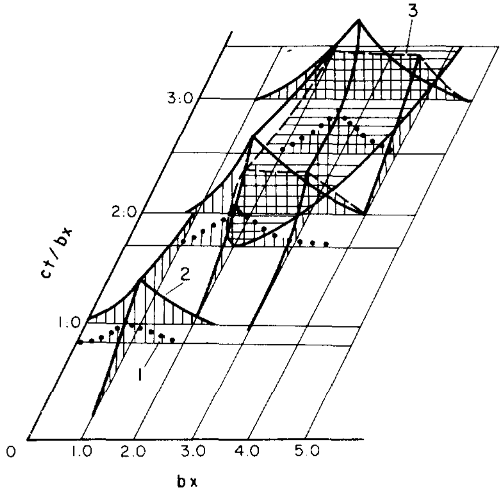


FIG. 2. The diagram of wave propagation due to temperature. 1—the modified thermoelastic wave. 2—the thermoelastic wave. 3—the thermoplastic wave.

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**Абстракт**—Изучены колебания и устойчивость тонкой, цилиндрической, круговой оболочки и трехслойной пластинки. Конструкция рассматривается как система с несколькими степенями свободы. Такая модель позволяет исследовать динамические свойства конструкции при действии сил и температуры. Получено приближенное решение для волновой задачи.